REPRESENTATION THEORY FINAL EXAMINATION

All representations and characters in the paper will be over the complex numbers $\mathbb{C}$.

Give details of all your calculations in questions 4 and 6 - the various conjugacy classes and their orders, the calculation of the characters and the description of the representations that correspond to them, and why each one is irreducible.

Total marks: 55

(1) Let G be a finite group, let χ_i denote the irreducible characters of G, let g_i denote representatives of conjugacy classes of G (say for $1 \leq i \leq r$). Let $X = (\chi_i(g_j))$ denote the $r \times r$ character table matrix of G, let $C(g_i)$ denote the conjugacy class of g_i in G

(a) What are the row orthogonality relations? Using them, show that the product matrix XCX^* is a scalar matrix, where $X^* = {}^t\overline{X}$ is the adjoint matrix of X, and C is the $r \times r$ diagonal matrix with diagonal entries $|C(g_i)|$ for $1 \leq i \leq r$.

(b) The column orthogonality relations are $\sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = \frac{|G|}{|C(g_i)|}\delta_{ij}$ for all $1 \leq i, j \leq r$. Can you deduce them from the previous part? (hint: if A, B are two square matrices such that AB is a nonzero scalar matrix, then show that $AB = BA$). (5+5=10 marks)

(2) Let G be a finite group with an odd number of elements. Prove that the only element of G which is conjugate to its inverse is the identity element of G. Using this or otherwise, show that given any $g \in G$ with $g \neq 1$, there exists an irreducible character χ of G such that $\chi(g)$ is not real (here χ is a complex valued function on G). (4+5 = 9 marks)

(3) Let V_n denote the standard representation of S_n (V_n is the irreducible representation of S_n of dimension $n-1$ whose direct sum with a copy of the trivial representation is the permutation representation of S_n on $\mathbb{C}^n$). Let W_n denote the one dimensional sign representation of S_n. For which values of $n \in \mathbb{N}$, are the two representations V_n and $V_n \otimes W_n$ isomorphic? (9 marks)

(4) Write down the character table of the symmetric group S_4. (9 marks)

(5) Find the irreducible components of the representations $V_4 \otimes V_4$ and $Sym^2(V_4)$ (where V_4 is the standard representation of S_4). (9 marks)

(6) Write down the character table for the dihedral group D_5 of order 10 (use: $cos(2\pi/5) = \frac{1-\sqrt{5}}{4}$ and $cos(4\pi/5) = -\frac{1+\sqrt{5}}{4}$) . (9 marks)